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Bernstein Type Inequalities for Quasipolynomials

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We prove a Bernstein type inequality for multivariate quasipolynomials and apply it to carry out the following results. (1) The evaluation of the uniform norm for a quasipolynomial on a convex body $V \subset \mathbb{R}^n$ by that on a measurable subset of V. (2) The estimate of the BMO-norm for a quasipolynomial in terms of its degree and exponential type. (3) The reverse Hölder inequality with a dimensionless constant.

Key Words: quasipolynomial; Bernstein inequality; analytic function; Chebyshev degree.

1. INTRODUCTION

1.1. The classical Bernstein inequality states that for a holomorphic polynomial $p \in \mathbb{C}[z_1, ..., z_n]$ of degree s

$$\max_{B_c(0,\, R)} |p(z)| \leqslant R^s \max_{B_c(0,\, 1)} |p(z)| \qquad (R > 1).$$

In the following $B_c(z, t) (\subset \mathbb{C}^n)$ stands for the complex Euclidean ball of radius t centered at $z = (z_1, ..., z_n)$.

The goal of this paper is to prove a similar inequality for a quasipolynomial and to obtain some new inequalities that result from it. For this we first recall several basic definitions.

Definition 1.1. Let $f_1, ..., f_k \in (\mathbb{C}^n)^*$ be a pairwise different set of complex linear functionals. A quasipolynomial with spectrum $sp(q) := \{f_1, ..., f_k\}$ is a finite sum

$$q = \sum_{i=1}^{k} p_i e^{f_i}, \tag{1.1}$$

where $p_i \in \mathbb{C}[z_1, ..., z_n]$ are holomorphic polynomials. The expression

$$m(q) := \sum_{i=1}^{k} (1 + \deg p_j)$$

is said to be the degree of q.

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We also introduce the *exponential* type of q by

$$\varepsilon(q) := \max_{1 \leqslant j \leqslant k} \max_{B_c(0,\,1)} |f_j|.$$

We now formulate the first basic result, an analog of Bernstein's inequality for quasipolynomials.

THEOREM 1.2. *For any quasipolynomial of degree m the inequality*

$$\max_{B_c(z,\,R)} |q| \leqslant C(\max\{1, \varepsilon(q)\})^{m-1} R^{m-1} e^{\varepsilon(q) R} \max_{B_c(z,\,1)} |q| \qquad (R > 1) \qquad (1.2)$$

holds with a constant $C = C(k, m)$, *that is, depending* only *on the parameters in the parenthesis.*

From the proof of this result it follows the next estimate of the constant:

$$C \leqslant m(2k)^m e^{2k-1} \prod_{i=1}^{k} (m_i)! < (km)^m e^{2k}.$$

Here $m_i := \deg p_i + 1$.

The following two examples show that the exponents $m-1$ and $\varepsilon(q) R$ in (1.2) are sharp.

(1) Let $q_\lambda(z) := (\sin \lambda z)^m$ be a univariate quasipolynomial with $\lambda > 0$ and an integer $m > 0$. Clearly the degree of q is $m+1$ and $\varepsilon(q_\lambda) = \lambda m$ while for a suitable constant $c = c(m) > 0$ and a sufficiently small λ and $R = 1/\sqrt{\lambda}$ we have

$$\max_{B_c(0,\,R)} |q_\lambda| \geqslant cR^m e^{\lambda m R} \max_{B_c(0,\,1)} |q_\lambda|.$$

(2) Let $h_\lambda(z) := (\sin \lambda z)^2 - \frac{\lambda z}{2} \sin 2\lambda z$, $\lambda > 0$, be a quasipolynomial of degree $m = 5$ (here $k = 3$) with $\varepsilon(h) = 2\lambda$. Then there is a constant $c' > 0$ such that for a sufficiently small λ and $R = 1/\sqrt{\lambda}$ we have

$$\max_{B_c(0,\,R)} |h_\lambda| \geqslant c' R^4 e^{2\lambda R} \max_{B_c(0,\,1)} |h_\lambda|.$$

This example also shows that one can not replace $m-1$ in inequality (1.2) by $k+d-1$, where $d := \max_{1 \leqslant j \leqslant k} \deg p_j$. In fact, in this case $k+d-1=3$.

REMARK 1.3. It is not difficult to prove that for a quasipolynomial $q=\sum_{i=1}^k p_i e^{f_i}$ with $d:=\max_{1\leqslant j\leqslant k} \deg p_j$ the inequality

$$\max_{B_c(0,R)} |q| \leqslant C(q)\, R^d e^{\varepsilon(q)\,R} \max_{B_c(0,1)} |q| \qquad (R>1)$$

holds. However, $\sup_q C(q)=\infty$ as the above mentioned examples show.

Remark 1.4. One possible application of Bernstein type inequalities is in estimating the number of zeros of univariate quasipolynomials in disks of different radii. This, in turn, leads to important results in the theory of transcendental numbers see, e.g. [Po], [P], [PS] and references there in.

1.2. Using the inequality of Theorem 1.2 we prove

THEOREM 1.5. *Let q be a quasipolynomial of degree m. Then there are absolute constants $c_1<15e^3$, $c_2<4e+1$ and $c_3<4e+1$ such that for any ball $B_c(z,r)\subset\mathbb{C}^n$, a real interval $I\subset B_c(z,r)$, and any measurable subset $\omega\subset I$ the inequality*

$$\sup_I |q| \leqslant \left(\frac{c_1\,|I|}{|\omega|}\right)^l \sup_\omega |q|$$

holds with $l=\log C+(m-1)\log(c_2\max\{1,\varepsilon(q)\})+c_3\varepsilon(q)\,r$. Here C is the constant of Theorem 1.2.

Remark 1.6. The best constant l for which the inequality of Theorem 1.5 is valid with $c_1=4$ is called according to [Br1, Def. 1.5] the *Chebyshev degree* of q in $B_c(z,r)$. Theorem 1.5 establishes an estimate of the Chebyshev degree.

We use this result to prove an inequality comparing the uniform norm of a quasipolynomial on a convex body V by its uniform norm on a measurable subset $\omega\subset V$. The sharp inequalities of this type for univariate polynomials were proved by Remez [R] in the 1930's and for the multivariate case by Brudnyi and Ganzburg [BG] in the 1970's (see also [E], [G], [Na] for various generalizations of Remez's inequalities and applications in Analysis). Below, $B(x,t)\subset\mathbb{R}^n(\subset\mathbb{C}^n)$ denotes a real Euclidean ball of radius t centered at x, and $|\omega|$ is the Lebesgue measure of $\omega\subset\mathbb{R}^n$.

THEOREM 1.7. *Let $V\subset\mathbb{R}^n$ be a convex body. Then for any quasipolynomial q defined on $\mathbb{C}^n$ and a measurable $\omega\subset V$ the inequality*

$$\max_V |q| \leqslant \left(\frac{c_1 n\,|V|}{|\omega|}\right)^\alpha \max_\omega |q|$$

holds with $\alpha = \log C + (m-1)\log(c_2 \max\{1, \varepsilon(q)\}) + (c_3/2)\,\varepsilon(q)\ \ diam(V)$ *and* c_1, c_2, c_3, C *as in Theorem* 1.5.

The following consequence of this theorem may be useful in considering compactness results for quasipolynomials.

Corollary 1.8. *Assume that* q_s, $s = 1, 2, \ldots$ *is a sequence of quasi-polynomials of degree* $\leqslant m$ *such that* $\lim_{s\to\infty} \varepsilon(q_s) = \infty$. *Then*

$$\limsup_{s\to\infty} \frac{1}{\varepsilon(q_s)\,|V|} \int_V \log \frac{\|q_s\|_V}{|q_s(x)|}\,dx \leqslant (c_3/2)\log(c_1 ne)\,diam(V).$$

Here $\|q\|_V := \max_V |q|$.

The next results were first established in [Br2] for analytic functions in terms of the Chebyshev degree. Theorem 1.5 allows us in the case of quasipolynomials to replace these estimates by more constructive ones. In order to formulate the corresponding theorems let us recall that a function $h: \mathbb{R}^s \to \mathbb{R}_+$ is *log-concave* if its support $K = \{x \in \mathbb{R}^s : h(x) > 0\}$ is convex and $\log h$ is a concave function on the support. Let μ_h be a measure on $\mathbb{R}^s$ with density h. For a convex body $V \subset \mathbb{R}^s$ we set

$$|V| := \mu_h(V), \qquad f_V := \exp\left(\frac{1}{|V|}\int_V \log|f|\,d\mu_h\right).$$

We assume without loss of generality that

$$\mu_h(V) = 1.$$

Theorem 1.9. *Let* $A^s \subset \mathbb{C}^n (\cong \mathbb{R}^{2n})$ *be an affine subspace of real dimension* s, $V \subset A^s$ *be an* s*-dimensional convex body, and* $h: A^s \to \mathbb{R}_+$ *be a log-concave function supported on* V. *There are absolute* (*i.e. independent of dimensions* s, n) *constants* c, $C > 0$ *such that for any quasipolynomial* q *defined on* $\mathbb{C}^n$

$$\text{(1)} \quad \mu_h\{x \in V : |q(x)| > t q_V\} \leqslant C \exp(-ct^{1/\alpha})$$

and (1.3)

$$\text{(2)} \quad \mu_h\{x \in V : |q(x)| \leqslant t q_V\} \leqslant C(ct)^{1/\alpha} (\log t)^{1/2}, \qquad t \leqslant e^{-1}.$$

Here α is the constant of Theorem 1.7.

COROLLARY 1.10. *Under the assumptions of Theorem* 1.9

$$\frac{1}{|V|}\int_V |q|^p \, d\mu_h \leqslant (cp\alpha)^{p\alpha}(q_V)^p \leqslant (cp\alpha)^{p\alpha}\left(\frac{1}{|V|}\int_V |q| \, d\mu_h\right)^p \qquad (p>1)$$

with an absolute constant $c>0$.

In particular, if $q_V \leqslant 1$, then the Orlicz norm of q defined by the Orlicz function $\phi(x) := \exp(x/\alpha) - 1$, $x \in \mathbb{R}_+$, on $(V, d\mu_h)$ is bounded by an absolute constant.

COROLLARY 1.11. *Under the assumptions of Theorem* 1.9

$$\frac{1}{|V|}\int_V |\log|q| - C_V(q)| \, d\mu_h \leqslant C\alpha.$$

Here $C>0$ *is an absolute constant and* $C_V(q) := \frac{1}{|V|}\int_V \log|q| \, d\mu_h$. *In particular, the BMO-norm of* $q|_V$ *is bounded by* $C\alpha$.

Remark 1.12. One can improve the estimate of the constant C established in Theorem 1.2. This leads to a sharper estimate of the constant α in all subsequent results. For example, in [P, Corol. 1] it was proved that the number of zeros of a univariate quasipolynomial q of degree at most m in $B_c(z, R) \subset \mathbb{C}$ does not exceed $4(m-1) + 3R\varepsilon(q)$. Using this estimate and an estimate of the Chebyshev degree of an analytic function by its local valency given in [Br1, Prop. 1.7] we obtain that $\alpha \leqslant c(4m + (3/2)\,\varepsilon(q)$ $diam(V))$. Here $c>0$ is an absolute constant that can be calculated explicitly.

2. PROOF OF THEOREM 1.2

We first prove the theorem under the assumption $\varepsilon(q) \leqslant 1$. Observe that there is a map $A(y) := g(y) + v$, $y \in \mathbb{C}^n$, where g is a unitary transform of $\mathbb{C}^n$ and $v \in \mathbb{C}^n$ sending $B_c(0, t)$ to $B_c(z, t)$. So if we prove the theorem for the quasipolynomial $q_1(y) := q(A(y))$ (which satisfies $deq\, q_1 = deq\, q$ and $\varepsilon(q_1) = \varepsilon(q)$) then going back to q we obtain the required inequality. Thus it suffices to prove the theorem for balls centered at 0.

Let l be a ray with the origin at 0 such that $\max_{B_c(0,R)} |q| = \max_{l \cap B_c(0,R)} |q|$. Arguing as above we may assume without loss of generality that l is a subset of the real axis x_1. Hence the required result follows from a similar one for univariate quasipolynomials:

Let $q(z)=\sum_{i=1}^{k} p_i(z)\, e^{a_i z}$, $a_i \in \mathbb{C}$, $p_i \in \mathbb{C}[z]$, $1 \leqslant i \leqslant k$, be a univariate quasipolynomial of degree $\leqslant m$ and $\varepsilon(q) := \max_{1 \leqslant i \leqslant k} |a_i| \leqslant 1$. Denote by $\mathbb{D} \subset \mathbb{C}$ the unit disk. Our goal is to prove that

$$\max_{0 \leqslant x \leqslant R} |q(x)| \leqslant C R^{m-1} e^{\varepsilon(q) R} \max_{z \in \mathbb{D}} |q(z)| \qquad (R > 1)$$

with $C = C(m, k, m_1, ..., m_k)$ and $m_i = \deg p_i + 1$. (Here we think of q as the restriction of the original quasipolynomial to the straight line containing l.)

Denote $a_s = x_s + i y_s$ and assume without loss of generality that $-1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k \leqslant 1$. Further, define quasipolynomials $\tilde{q}_s$ and q_s, $1 \leqslant s \leqslant k$, as

$$\tilde{q}_1(z) = q(z)\, e^{-a_1 z},$$

$$q_1(z) = \tilde{q}_1^{(m_1)}(z) = \sum_{i=2}^{k} p_{i1}(z)\, e^{(a_i - a_1) z};$$

$$\tilde{q}_s(z) = q_{s-1}(z)\, e^{-(a_s - a_{s-1}) z},$$

$$q_s(z) = \tilde{q}_s^{(m_s)}(z) = \sum_{i=s+1}^{k} p_{is}(z)\, e^{(a_i - a_s) z} \qquad (s > 1).$$

Here $\deg p_{is} \leqslant \deg p_i$; $\tilde{q}_k$ is a polynomial of degree $\leqslant \deg p_k$ and $q_k = 0$.

Hereafter $\mathbb{D}_t$ denotes the disk of radius t centered at 0. In the next step we estimate $\max_{z \in \mathbb{D}_{1-i/2k}} |q_i(z)|$. Let $A := \max_{z \in \mathbb{D}} |q(z)|$. Then

$$|\tilde{q}_1(z)| = |q(z)\, e^{-a_1 z}| \leqslant A e^{|a_1|} \qquad (z \in \mathbb{D}).$$

To estimate $\max_{z \in \mathbb{D}_{1-1/2k}} |q_1(z)|$ we apply Cauchy's inequalities for the derivatives of a holomorphic function

$$\max_{z \in \mathbb{D}_{1-1/2k}} |q_1(z)| \leqslant \max_{z \in \mathbb{D}_{1-1/2k}} \left\{ \frac{1}{2\pi} \left| \int_{|z|=1} \frac{(m_1)!\, \tilde{q}_1(y)}{(y-z)^{m_1+1}}\, dy \right| \right\}$$
$$\leqslant (2k)^{m_1+1}\, (m_1)!\, A e^{|a_1|} (:= A_1).$$

From these inequalities we also have

$$|\tilde{q}_1^{(l)}(0)| \leqslant A_1, \qquad 0 \leqslant l \leqslant m_1.$$

Continuing by induction we obtain for $1 < s \leqslant k$

$$\max_{z \in \mathbb{D}_{1-(s-1)/2k}} |\tilde{q}_s(z)| \leqslant A_{s-1} e^{|a_s - a_{s-1}|}$$

$$\max_{z \in \mathbb{D}_{1-s/2k}} |q_s(z)| \leqslant (2k)^{m_s+1}\, (m_s)!\, A_{s-1}\, e^{|a_s - a_{s-1}|} (:= A_s) \tag{2.1}$$

$$|\tilde{q}_s^{(l)}(0)| \leqslant A_s, \qquad 0 \leqslant l \leqslant m_s.$$

Finally,

$$\max_{z \in \mathbb{D}_{1/2}} |\tilde{q}_k(z)| \leqslant A(2k)^{m-m_k-1} \cdot e^{|a_1| + \sum_{i=1}^{k-1} |a_{i+1}-a_i|} \cdot \prod_{i=1}^{k-1} (m_i)! \ (:= A'_k).$$

Note also that

$$A < A_1 < \cdots < A_{k-1} \leqslant A'_k.$$

Since $\tilde{q}_k$ is a polynomial of degree $\leqslant m_k - 1$, the Bernstein inequality for polynomials implies that

$$\max_{z \in \mathbb{D}_t} |\tilde{q}_k(z)| \leqslant (2R)^{m_k-1} A'_k, \qquad R \geqslant 1/2.$$

From the last estimate it follows that for $x \in [0, R]$

$$|q_{k-1}(x)| \leqslant C_1 R^{m_k-1} e^{R(x_k - x_{k-1})} \tag{2.2}$$

with $C_1 = 2^{m_k-1} A'_k$. Integrating this inequality and using (2.1) we get (for $R > 1$ and $x \in [0, R]$)

$$|\tilde{q}_{k-1}^{(m_{k-1}-1)}(x)| \leqslant |\tilde{q}_{k-1}^{(m_{k-1}-1)}(0)| + \int_0^x |q_{k-1}(t)|\, dt \leqslant 2C_1 R^{m_k} e^{R(x_k - x_{k-1})}. \tag{2.3}$$

Repeating this procedure after $m_{k-1} - 1$ steps we obtain

$$|\tilde{q}_{k-1}(x)| \leqslant (m_{k-1}+1)\, C_1 R^{m_k + m_{k-1} - 1} e^{R(x_k - x_{k-1})} \qquad (x \in [0, R]).$$

Now we can apply the very same arguments to $\tilde{q}_{k-1}$ to estimate $\tilde{q}_{k-2}$ etc. Finally, we have

$$\max_{x \in [0, R]} |q(x)| \leqslant \left(1 + \sum_{i=1}^{k-1} m_i\right) C_1 R^{m-1} e^{|a_k| R} \leqslant C R^{m-1} e^{\varepsilon R} \max_{z \in \mathbb{D}} |q(z)| \tag{2.4}$$

with

$$C = m(2k)^m \cdot e^{l(q)} \cdot \prod_{i=1}^{k-1} (m_i)! < 2(km)^m e^{l(q)}. \tag{2.5}$$

Here $l(q) := |a_1| + \sum_{i=1}^{k-1} |a_{i+1} - a_i| \leqslant 2k - 1$ for generic $a_1, \dots, a_k$ and $l(q) \leqslant 3$ if these points belong to a straight line. Finally, note that $\prod_{i=1}^{k} (m_i)!$ for the restriction $q|_l$ is less then or equal to the similar expression for the quasipolynomial q itself.

Thus we have proved the proposition under the assumption $\varepsilon(q) \leqslant 1$. Let us consider the case $\varepsilon(q) > 1$.

For the quasipolynomial q let us define

$$q_1(z) := q(z/\varepsilon(q)).$$

Clearly q_1 is a quasipolynomial with the same characteristics as q and with $\varepsilon(q_1) = 1$. Thus according to the inequality of the theorem for $\varepsilon(q) \leqslant 1$ we have

$$\begin{aligned}\max_{B_c(0,\,R)} |q| = \max_{B_c(0,\,\varepsilon(q)\,R)} |q_1| &\leqslant C(\varepsilon(q)\,R)^{m-1}\, e^{\varepsilon(q)\,R} \max_{B_c(0,\,1)} |q_1| \\ &\leqslant C(\varepsilon(q)\,R)^{m-1}\, e^{\varepsilon(q)\,R} \max_{B_c((0,\,1)} |q|.\end{aligned}$$

The proof of Theorem 1.2 is complete. ∎

3. PROOF OF THEOREMS 1.5 AND 1.7 AND COROLLARY 1.8

Proof of Theorem 1.5. In the proof we use the result proved in Levin's book [L, p. 21]. A slightly different proof can be done by the method presented in [Br1].

LEMMA 3.1. *Let $f(z)$ be a holomorphic function on $\mathbb{D}_{2eR}$, $f(0) = 1$ and η be a positive number $\leqslant \frac{3e}{2}$. Then there is a set of disks $\{D_i\}$ with $\sum_i r_i \leqslant 4\eta R$, where r_i is radius of D_i such that*

$$\log |f(z)| > -H(\eta) \log \max_{\mathbb{D}_{2eR}} |f|$$

for any $z \in \mathbb{D}_R \backslash (\bigcup_i D_i)$. Here $H(\eta) = 2 + \log \frac{3e}{2\eta}$.

Let q be a quasipolynomial of degree m and I be a real interval in $B_c(z, r)$. Let l_c be a complex straight line containing I. Without loss of generality we may consider I as an interval $[-a, a]$ on the real axis x_1 such that $a < r$. Then l_c coincides with $\mathbb{C}$. We set $q_1 := q|_{l_c}$. Let $\omega \subset I$ be a measurable subset. Consider two disks $\mathbb{D}_a \subset \mathbb{D}_{2ea} \subset \mathbb{C}$. Let $w \in \mathbb{D}_a$ be such that

$$\max_{\mathbb{D}_a} |q_1| = |q_1(w)|.$$

Further consider disks $D(w, 2a) \subset D(w, 4ea) \subset l_c$ centered at w with radii $2a$ and $4ea$, respectively. Observe that $D(w, 4ea) \subset \mathbb{D}_{(2e+1/2)\,2a}$. We apply

Lemma 3.1 to $D(w, 2a) \subset D(w, 4ea)$ and $f = q_1/|q_1(w)|$ with $\eta = |\omega|/20a$. Then we obtain

$$\log|f(z)| > -H(\eta) \log \max_{D(w, 4ea)} |f|$$

for any $z \in D(w, 2a) \backslash (\bigcup_i D_i)$ where $\{D_i\}$ is a set of disks with $\sum_i r_i \leqslant 2|\omega|/5$, and r_i is radius of D_i. Here $H(\eta) = 2 + \log \frac{30ea}{|\omega|}$. Note that $\bigcup_i D_i$ can not cover ω because of the choice of η. Therefore there is a point $x_0 \in \omega$ for which the last inequality holds, that is,

$$-\left(2 + \log \frac{30ea}{|\omega|}\right) \log \max_{D(w, 4ea)} \frac{|q_1|}{|q_1(w)|} \leqslant \log \frac{|q_1(x_0)|}{|q_1(w)|} \leqslant \log \max_{\omega} \frac{|q_1|}{|q_1(w)|}. \qquad (3.1)$$

Applying now inequality of Theorem 1.2 to q_1 and $\mathbb{D}_a \subset \mathbb{D}_{(2e+1/2)\,2a}$ and taking into account that

$$\max_{D(w, 4ea)} |q_1| \leqslant \max_{\mathbb{D}_{(2e+1/2)\,2a}} |q_1| \qquad \text{and} \qquad \max_I |q_1| \leqslant \max_{\mathbb{D}_a} |q_1|$$

we obtain from (3.1)

$$-\left(2 + \log \frac{15e\,|I|}{|\omega|}\right) \log[\,C(\max\{1, \varepsilon(q_1)\}(4e+1))^{m-1}\, e^{\varepsilon(q_1)(2e+1/2)\,|I|}\,]$$
$$\leqslant \log \frac{\max_\omega |q_1|}{\max_I |q_1|}.$$

Taking the exponent in both sides of this inequality and using the inequalities $\varepsilon(q_1) \leqslant \varepsilon(q)$, $|I| \leqslant 2r$ we get

$$\max_I |q| \leqslant \left(\frac{c_1\,|I|}{|\omega|}\right)^l \max_\omega |q|$$

with $l = \log C + (m-1)\log(c_2 \max\{1, \varepsilon(q)\}) + c_3 \varepsilon(q)\, r$; $c_1 = 15e^3$, $c_2 = \log(4e+1)$ and $c_3 = 4e+1$.

The proof of Theorem 1.5 is complete. ∎

Proof of Theorem 1.7. Let $V \subset \mathbb{R}^n$ be a convex body and $\omega \subset V$ be a measurable subset. Choose a point $x \in V$ such that $|q(x)| = \max_V |q|$. (Without loss of generality we may assume that x is an interior point of V; for otherwise, we apply the arguments below to an interior point x_ε, $\varepsilon > 0$, such that $|q(x_\varepsilon)| > \max_V |q| - \varepsilon$ and then take the limit as $\varepsilon \to 0$.) According to Lemma 3 of [BG] there is a ray with origin at x such that

$$\frac{mes_1(l \cap V)}{mes_1(l \cap \omega)} \leqslant \frac{n\,|V|}{|\omega|}. \tag{3.2}$$

We set $I := l \cap V$, $\omega_1 := l \cap \omega$ and apply the inequalities of Theorem 1.5 and (3.2) to this pair. Then we get

$$\sup_V |q| = \sup_I |q| \leqslant \left(\frac{c_1\,|I|}{|\omega_1|}\right)^\alpha \sup_{\omega_1} |q| \leqslant \left(\frac{c_1 n\,|V|}{|\omega|}\right)^\alpha \sup_\omega |q|.$$

Here $\alpha = \log C + (m-1)\log(c_2 \max\{1, \varepsilon(q)\}) + (c_3/2)\,\varepsilon(q)\,diam(V)$ and c_1, c_2, c_3, C as in Theorem 1.5. ∎

Proof of Corollary 1.8. Let V be a convex body and q be a quasi-polynomial of degree m. For the distribution function $D_q(t) := mes\{x \in V; |q(x)| \leqslant t\}$ the inequality of Theorem 1.7 acquires the form

$$D_q(t) \leqslant c_1 n\,|V| \left(\frac{t}{\|q\|_V}\right)^{1/\alpha}.$$

Let $q_*(t) = \inf\{s : D_q(s) \geqslant t\}$. Then from the above inequality for D_q we obtain

$$\int_V \log \frac{\|q\|_V}{|q(x)|}\,dx = \int_0^{|V|} \log \frac{\|q\|_V}{q_*(t)}\,dt \leqslant \int_0^{|V|} \log \left(\frac{c_1 n\,|V|}{s}\right)^\alpha ds = \alpha \log(c_1 en).$$

Then the required result trivially follows from this inequality. ∎

4. PROOF OF THEOREM 1.9 AND COROLLARIES 1.10 AND 1.11

Proof of Theorem 1.9. Our main tool is a remarkable result of Kannan, Lovász and Simonovits ([KLS, Cor. 2.21]) which reduces the estimation of a multidimensional integral to corresponding one-dimensional ones. Using this we establish the following basic inequality which gives Theorem 1.9 as a simple consequence.

Proposition 4.1. *Let q, V, $r > 1$, μ_h and α be as in Theorem* 1.9. *Then*

$$\left(\frac{1}{|V|}\int_V |q|^m\,d\mu_h\right)^n \left(\frac{1}{|V|}\int_V |q|^{-p}\,d\mu_h\right)^r \leqslant (2e)^{n+r}\,c_1^{(mn+pr)\,\alpha}\,\frac{\Gamma(\alpha+1)^n}{(1-p\alpha)^r}$$

provided m, n, p, $r>0$ *satisfy*

$$mn=pr, \qquad p<\frac{1}{\alpha}.$$

Here, as usual, $\Gamma(x):=\int_0^\infty t^{x-1}e^{-t}\,dt$ *and* $c_1<15e^3$ *is the constant of Theorem* 1.5.

First, we formulate several results used in the proof of the proposition. We introduce the following definition (see [KLS]). By an *exponential needle* we mean a segment $I=[x, y]$ in $\mathbb{R}^n$, together with a real constant γ. If (E, γ) is an exponential needle and f is an integrable function defined on I, then we set

$$\int_E f=\int_0^{|y-x|} f(x+tu)\, e^{\gamma t}\,dt,$$

where $u=(1/|y-x|)(y-x)$.

THEOREM 4.2 [KLS]. *Let* f_1, f_2, f_3, f_4 *be four nonnegative continuous functions defined on* $\mathbb{R}^n$, *and* a, $b>0$. *Then the following are equivalent*:

(a) *For every log-concave function* F *defined on* $\mathbb{R}^n$ *with compact support*,

$$\left(\int_{\mathbb{R}^n} F(t)\, f_1(t)\,dt\right)^a \left(\int_{\mathbb{R}^n} F(t)\, f_2(t)\,dt\right)^b \leqslant \left(\int_{\mathbb{R}^n} F(t)\, f_3(t)\,dt\right)^a \left(\int_{\mathbb{R}^n} F(t)\, f_4(t)\,dt\right)^b.$$

(b) *For every exponential needle* E

$$\left(\int_E f_1\right)^a \left(\int_E f_2\right)^b \leqslant \left(\int_E f_3\right)^a \left(\int_E f_4\right)^b.$$

Remark 4.3. The above theorem is also valid for nonnegative f_1, f_2, f_3, f_4 such that f_1, f_2 are the limits of monotone increasing sequences of continuous functions defined on $\mathbb{R}^n$ and f_3, f_4 are the limits of monotone decreasing sequences of continuous functions defined on $\mathbb{R}^n$ (see Remark 2.3 in [KLS]). In particular, we can apply this theorem in the case of K a closed convex body, f_1, f_2 nonnegative continuous functions defined on K which are 0 outside K and f_3, f_4 nonnegative functions which are constant on K and 0 outside.

We also use the next distributional inequality that follows directly from the inequality of Theorem 1.5.

Let $V \subset A^s \subset \mathbb{C}^n$ be an s-dimensional convex body, $I \subset V$ be a real segment and q be a quasipolynomial. For the distribution function $D_{q_I}(t) := |\{x \in I : |q(x)| \leqslant t\}|$ (with respect to the usual Lebesgue measure on I) let us define $(q_I)_*(t) = \inf\{y : D_{q_I}(y) \geqslant t\}$. Then

$$(q_I)_*(t) \geqslant \left(\frac{t}{c_1 \, |I|}\right)^{\alpha} \sup_V |q| \tag{4.1}$$

with $\alpha = \log C + (m-1) \log(c_2 \max\{1, \varepsilon(q)\}) + (c_3/2)\, \varepsilon(q)\, diam(V)$, $c_1 < 15e^3$ (cf. Theorem 1.7).

Proof of Proposition 4.1. Let q be a quasipolynomial and $I \subset V$ be a real interval. Then the functions $q_\varepsilon := (|q| + \varepsilon)|_I$, $\varepsilon > 0$, and $q_{\varepsilon, a, b}(t) = q_\varepsilon(at + b)$, $t \in I$, a, $b \in \mathbb{R}$, also satisfy the inequality of Theorem 1.5. We must apply the KLS theorem to functions $q_1 := (|q| + \varepsilon)^m$, $q_2 := (|q| + \varepsilon)^{-p}$ (continuous on V) and $q_3 := 2e \cdot c_1^{m\alpha} \Gamma(m\alpha + 1)$, $q_4 := 2e \cdot c_1^{p\alpha}/(1 - p\alpha)$ on V and 0 outside V and then take the limit when $\varepsilon \to 0$. To avoid abuse of notation and because our estimates below do not depend on ε we may assume without loss of generality that $|q|$ itself has no zeros on V.

According to the KLS theorem and Remark 4.3 the proposition follows from the inequality

$$\left(\int_E |q|^m\right)^n \left(\int_E |q|^{-p}\right)^r \leqslant (2e)^{n+r}\, c_1^{(mn+pr)\,\alpha}\, \frac{\Gamma(m\alpha + 1)^n}{(1 - p\alpha)^r} \left(\int_E 1\right)^{n+r}$$

for an exponential needle $E \subset V$. Making an affine change of variables in the above integrals we reduce the problem to the following inequality

$$\left(\int_0^s |\tilde{q}(x)|^m e^{-x}\, dx\right)^n \left(\int_0^s |\tilde{q}(x)|^{-p} e^{-x}\, dx\right)^r$$
$$\leqslant (2e)^{n+r}\, c_1^{(mn+pr)\,\alpha}\, \frac{\Gamma(m\alpha + 1)^n}{(1 - p\alpha)^r}\, (1 - e^{-s})^{n+r}.$$

Here $\tilde{q}$ is a function obtained from q by this change of variables. As we already mentioned $\tilde{q}$ satisfies inequality of Theorem 1.5 with α instead of l. Below we denote $\|\tilde{q}\| := \sup_I |\tilde{q}|$.

First, let $0 \leqslant s \leqslant 1$. Then

$$
\begin{aligned}
&\left(\int_0^s |\tilde{q}(x)|^m e^{-x}\, dx\right)^n \left(\int_0^s |\tilde{q}(x)|^{-p} e^{-x}\, dx\right)^r \\
&\leqslant \left(\int_0^s \left(\frac{|\tilde{q}(x)|}{\|\tilde{q}\|_{[0,s]}}\right)^m dx\right)^n \left(\int_0^s \left(\frac{\|\tilde{q}\|_{[0,s]}}{|\tilde{q}(x)|}\right)^p dx\right)^r \\
&\leqslant s^n \left(\int_0^s \left(\frac{\|\tilde{q}\|_{[0,s]}}{\tilde{q}_*(t)}\right)^p dt\right)^r \\
&\leqslant s^n \left(s \int_0^1 \left(\frac{c_1}{t}\right)^{p\alpha} dt\right)^r \leqslant c_1^{pr\alpha} s^{n+r} \left(\frac{1}{1-p\alpha}\right)^r \\
&\leqslant c_1^{pr\alpha} (2(1-e^{-s}))^{n+r} \left(\frac{1}{1-p\alpha}\right)^r.
\end{aligned}
$$

Here we applied inequality (4.1) to the lower distribution function $\tilde{q}_*$ of $\tilde{q}$ and used the inequality $1-e^{-s} > s/2$ for $0 < s \leqslant 1$. Observe that the obtained constant is even less than the required one.

Assume now that $s > 1$. We estimate each of the two factors of the given expression. Without loss of generality we may assume that s is an integer. Then

$$
\begin{aligned}
\int_0^s |\tilde{q}(x)|^m e^{-x}\, dx &= \sum_{i=0}^{s-1} \int_i^{i+1} |\tilde{q}(x)|^m e^{-x}\, dx \\
&\leqslant \sum_{i=0}^{s-1} \left(\int_i^{i+1} |\tilde{q}(x)|^m\, dx\right) e^{-i} \\
&\leqslant \sum_{i=0}^{s-1} \left(\int_i^{i+1} \left(\frac{|\tilde{q}(x)|}{\|\tilde{q}\|_{[i,i+1]}} \|\tilde{q}\|_{[i,i+1]}\right)^m dx\right) e^{-i} \\
&\leqslant \sum_{i=0}^{s-1} \|\tilde{q}\|_{[0,i+1]}^m e^{-i} \\
&\leqslant \sum_{i=0}^{\infty} (c_1(i+1))^{m\alpha} e^{-i} \|\tilde{q}\|_{[0,1]}^m \\
&\leqslant c_1^{m\alpha} e \int_0^\infty x^{m\alpha} e^{-x}\, dx \, \|\tilde{q}\|_{[0,1]}^m \\
&= c_1^{m\alpha} e \Gamma(m\alpha+1) \, \|\tilde{q}\|_{[0,1]}^m.
\end{aligned}
$$

We used here inequality of Theorem 1.5 to estimate $\sup_{[0,\,i+1]} |\tilde{q}|$ by $\sup_{[0,\,1]} |\tilde{q}|$. Similarly,

$$\begin{aligned}
\int_0^s |\tilde{q}(x)|^{-p}\, e^{-x}\, dx &\leqslant \sum_{i=0}^{s-1} \left(\int_i^{i+1} |\tilde{q}(x)|^{-p}\, dx \right) e^{-i} \\
&\leqslant \sum_{i=0}^{s-1} \left(\int_0^{i+1} \left(\frac{\|\tilde{q}\|_{[0,\,i+1]}}{|\tilde{q}(x)|} \right)^p \frac{1}{\|\tilde{q}\|^p_{[0,\,i+1]}}\, dx \right) e^{-i} \\
&\leqslant \sum_{i=0}^{s-1} \left(\int_0^{i+1} \left(\frac{\|\tilde{q}\|_{[0,\,i+1]}}{\tilde{q}_*(t)} \right)^p \frac{1}{\|\tilde{q}\|^p_{[0,\,1]}}\, dt \right) e^{-i} \\
&\leqslant \left(\sum_0^{s-1} \int_0^{i+1} \left(\frac{c_1(i+1)}{t} \right)^{p\alpha} \frac{1}{\|\tilde{q}\|^p_{[0,\,1]}}\, dt \right) e^{-i} \\
&\leqslant \sum_{i=0}^{s-1} \frac{c_1^{p\alpha}(i+1)}{1-p\alpha} \frac{1}{\|\tilde{q}\|^p_{[0,\,1]}}\, e^{-i} \leqslant \frac{e c_1^{p\alpha}}{1-p\alpha} \frac{1}{\|\tilde{q}\|^p_{[0,\,1]}}.
\end{aligned}$$

Using that $pr = mn$ and $1 - e^{-s} \geqslant 1/2$ for $s \geqslant 1$ we get from these inequalities

$$\begin{aligned}
&\left(\int_0^s |\tilde{q}(x)|^m\, e^{-x}\, dx \right)^n \left(\int_0^s |\tilde{q}(x)|^{-p}\, e^{-x}\, dx \right)^r \\
&\qquad \leqslant c_1^{(mn+pr)\,\alpha} \cdot (2e)^{n+r} \frac{\Gamma(m\alpha+1)^n}{(1-p\alpha)^r} (1-e^{-s})^{n+r}.
\end{aligned}$$

This completes the proof of the proposition. ∎

We proceed to the proof of Theorem 1.9.

(1) We apply Proposition 4.1 to q with $n=1$, $p=1/(2\alpha)$, $r=2m$ and with m/α instead of m. Here m is a positive integer. Below, denote $g := |q|^{1/\alpha}$. Assume without loss of generality that $g_V = 1$ and set $E_w := \{x \in V : g(x) > w\}$, $|E_w| := \mu_h(E_w)$. Then from Proposition 4.1 we obtain

$$\begin{aligned}
&w^m\, |E_w| \left(\int_V g^{-1/2}\, d\mu_h \right)^{2m} \\
&\qquad \leqslant \left(\int_V g^m\, d\mu_h \right) \left(\int_V g^{-1/2}\, d\mu_h \right)^{2m} \leqslant c_1^{2m} (2e)^{2m+1}\, 2^{2m} (m!)
\end{aligned}$$

which is equivalent to (because $c_1 < 15e^3$)

$$\begin{aligned}
w^m\,|E_w| &\leqslant \frac{c_1^{2m}2^{4m+1}e^{2m+1}(m!)}{(\int_V g^{-1/2}\,d\mu_h)^{2m}}\\
&\leqslant 2^{12m+1}e^{8m+1}(m!)\exp\left(-2m\log\left(\int_V g^{-1/2}\,d\mu_h\right)\right)\\
&\leqslant 2^{12m+1}e^{8m+1}(m!)(g_V)^m = 2^{12m+1}e^{8m+1}(m!) \leqslant e^{20m}(m!).
\end{aligned} \tag{4.2}$$

We used here Jensen's inequality

$$\int_V g^{-1/2}\,d\mu_h \geqslant \exp\left(\frac{-1}{2}\int_V \log g\,d\mu_h\right).$$

Since $|V| = 1$, we also have

$$|E_w| \leqslant 1.$$

Dividing both sides of (4.2) by $e^{21m}(m!)$ and summing over m from 0 to ∞ we get

$$\exp(w/e^{21})\,|E_w| \leqslant 2$$

or

$$|E_w| \leqslant 2\exp(-w/e^{21}).$$

Since $g := |q|^{1/\alpha}$, the required inequality follows from here.

This proves part (1).

(2) Recall that $C_V(q) := \frac{1}{|V|}\int_V \log|q|\,d\mu_h$. We will estimate the measure $|F_\gamma| := \mu_h(F_\gamma)$ of the set $F_\gamma := \{x \in V : |\log|q| - C_V(q)| \geqslant \gamma\}$, $\gamma \geqslant 1$. We apply Proposition 4.1 to q with $m = p = (1-1/\gamma)/\alpha$, $n = q = 1$. Then we have

$$\begin{aligned}
e^{(2\gamma(1-1/\gamma)/\alpha)}\,|F_\gamma|^2 &\leqslant \left(\int_V e^{((1-1/\gamma)/\alpha)(\log|q| - C_V(q))}\,d\mu_h\right)\\
&\quad\times\left(\int_V e^{(-(1-1/\gamma)/\alpha)(\log|q| - C_V(q))}\,d\mu_h\right)\\
&\leqslant c_1^{2(1-1/\gamma)}(2e)^2\,\frac{\Gamma(2-1/\gamma)}{1-(1-1/\gamma)} \leqslant 2^9 e^8\gamma.
\end{aligned}$$

Hence

$$|F_\gamma| \leqslant 2^5 e^{4+1/\alpha} e^{-\gamma/\alpha}\gamma^{1/2}. \tag{4.3}$$

This, in particular, gives an estimate of $\mu_h\{x \in V : \log |q| - C_V(q) \leqslant -\gamma\}$ which, in turn, gives the required result

$$\mu_h\{x \in V : |q(x)| \leqslant tq_V\} \leqslant 2^5 e^4 (et)^{1/\alpha} (\log t)^{1/2}, \qquad t \leqslant e^{-1},$$

with $t = e^{-\gamma}$.

The proof of Theorem 1.9 is complete. ∎

Proofs of corollaries. Corollary 1.10 follows directly by integration of inequality (1) of Theorem 1.9 and Corollary 1.11 is a simple consequence of inequality (4.3). ∎

REFERENCES

[BG] Yu. Brudnyi and M. Ganzburg, On the extremal problem for polynomials of n-variables, *Math. USSR Izv.* **37** (1973), 344–355.

[Br1] A. Brudnyi, On local behavior of analytic functions, *J. Funct. Anal.* **169** (1999), 481–493.

[Br2] A. Brudnyi, The distribution of values of analytic functions on convex sets, preprint, 2000.

[E] T. Erdélyi, Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc. (2)* **45** (1992), 255–264.

[G] M. Ganzburg, Polynomial inequalities on measurable sets and their applications II. Weighted measures, *J. Approx. Theory* **106** (2000), 77–109.

[KLS] R. Kannan, L. Lovász, and M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, *Discrete Comput. Geom.* **13** (1995), 541–559.

[L] B. Ya. Levin, "Distribution of Zeros of Entire Functions," Amer. Math. Soc., Providence, RI, 1986.

[Na] F. Nazarov, Local estimates of exponential polynomials and their applications to inequalities of uncertainty principle type, *St. Petersburg Math. J.* **5** (1994), 663–717.

[P] A. J. van der Poorten, On the number of zeros of functions, *Enseign. Math. (2)* **23** (1977), 19–38.

[Po] G. Pólya, Geometrisches über the Verteilung der Nullstellen gewisser ganzer transzendenter Funktionen, *Münschener Sitzungsberichte* **50** (1920), 285–290.

[PS] A. J. van der Poorten and I. E. Shparlinski, On the number of zeros of exponential polynomials and related questions, *Bull. Austral. Math. Soc.* **46** (1992), 401–412.

[R] E. Remez, Sur une propiérté extrémale des polynômes de Tchebychef, *Zap. Nauk-Doslid. Inst. Math. Meh., Harkiv, Mat. Tovar.* **13**, No. 4 (1936), 93–95.